

On the Structure of the Set of Correlated Equilibria in two-by-two Bimatrix Games

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May 1999

Abstract

The paper studies the structure of the set of correlated equilibria for 2×2 -bimatrix games. We find that the extreme points of the (convex) set of correlated equilibria can be determined very easily from the Nash equilibria of the game.

JEL Classification: C72.

Keywords: Correlated equilibrium; bimatrix game.

1 Introduction.

In certain classes of strategic games the players have *partially common interests* and they may fear that ‘just playing a Nash equilibrium’ does not do justice to the common interests. In such games it may be wise to introduce a cooperative pre-play meeting to coordinate the actions of the different players. The concept of *correlated equilibria* is based on this idea (see Auman (1974) and (1987)). It gives a method to coordinate the actions of the players before the game is played.

The idea is the following: in the pre-play meeting the players agree upon a (*finite*) *probability space* Ω *with probability measure* p and *signalling functions* $x_i: \Omega \rightarrow T_i$, one for each player.

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If the game is played, a chance mechanism with probability distribution p (out of the reach of any player) determines a point $\omega \in \Omega$ and each player i get the information $x_i(\omega) \in T_i$. Now each player i chooses an action a_i in his action space A_i and the payoffs $u_i(a_1, \dots, a_n)$ follow. The difference with playing the original strategic game $(\{A_i, u_i\}_{i \in N})$ is that the players can choose strategies $f_i: T_i \rightarrow A_i$, the players can react to the signal they get. During the pre-play meeting the players transformed the strategic game $(\{A_i, u_i\}_{i \in N})$ into a *Bayesian game* $(\Omega, p, \{x_i, T_i\}_{i \in N}, \{A_i, u_i\}_{i \in N})$.

A Bayesian equilibrium in the extended game is called a *correlated equilibrium* of the strategic game $(\{A_i, u_i\}_{i \in N})$. Note that the signal is costless and is only used to ‘coordinate’ the actions.

Suppose that $\{f_i\}_{i \in N}$ is a Bayesian equilibrium in the game

$$(\Omega, p, \{x_i, T_i\}_{i \in N}, \{A_i, u_i\}_{i \in N}),$$

then we can introduce a new set $\bar{\Omega} := A_1 \times \dots \times A_n$ and a new probability measure \bar{p} on $\bar{\Omega}$ defined by $\bar{p}(a_1, \dots, a_n) := p[\{\omega \in \Omega : f_i \circ x_i(\omega) = a_i\}]$. We also introduce new signalling functions, namely $\bar{x}_i(a_1, \dots, a_n) := a_i$.

So, in the pre-play meeting the players agree upon a chance distribution over the strategy profiles, before the game is played a strategy profile is drawn at random and each player gets as a signal his component of this strategy profile a_i , the ‘strategy he is supposed to play’. ‘Following the advice’, i.e. playing a_i if you are told to do so, is a Bayesian equilibrium in the new situation and generates the same outcome, the same chance distribution over $A = A_1 \times \dots \times A_n$, as the old Bayesian equilibrium $\{f_i\}$ did (see Osborne and Rubinstein (1994) for a proof). As the chance distribution over A is the only thing that matters, the players do not need fancy chance mechanisms (Ω, p) and signalling functions $\{x_i\}$, they can get the same outcome by taking $\Omega = A$ and signalling functions $x_i(a) = a_i$. What they have to discuss is the probability distribution p on A . So, the set of *correlated equilibria* consists of all probability measures $p \in \Delta(A)$ for which the reaction functions ‘following the advice you get’ form a Bayesian equilibrium. Note that ‘following the advice you get’ generates the probability distribution p as *outcome* and payoffs $\sum_{a \in A} p(a) u_i(a)$ for each player i . So, after all the discussion during the pre-play meeting was about the outcome.

After the previous discussion we are left with the following situation: during the cooperative phase of the game the players (try to) agree upon an element $p \in \Delta(A)$. During the game each player gets the advice to play a certain action. If they do so, the probability distribution p is generated and the question remains if it is wise to follow the instructions. Is the situation *self-enforcing* in the sense that no unilateral deviation is a better action for the deviating player?

In this paper we study the set of correlated equilibria for the most simple situation, for 2×2 -bimatrix games. We find a very easy way to derive the set of correlated equilibria from the set of Nash equilibria.

For these games there are two well-known facts:

- (i) the set of correlated equilibria is a convex polyhedron,
- (ii) Nash equilibria ‘are’ correlated equilibria.

From these facts follows that the set of correlated equilibria contains the convex hull of the set of Nash equilibria. So we are interested in extreme points of the set of correlated equilibria *that are not Nash equilibria*.

We prove that such points exist if and only if the 2×2 -bimatrix game has three isolated Nash equilibria, two pure and one mixed Nash equilibrium (it is a game like the ‘Battle of the Sexes’) and that the extreme points of the set of correlated equilibria can be found from the coordinates of the mixed equilibrium.

2 Correlated equilibria for bimatrix games.

Let (A, B) be the payoff matrices of a bimatrix game of size $m \times n$. Let $Z = (z_{ij}) \geq 0$ be a probability vector on the entries (i, j) . So, $\sum_{ij} z_{ij} = 1$. We first write down the conditions that Z must satisfy to be a *self-enforcing* solution. If the signal e_i is given, the conditional expected payoff of playing e_i must be at least as large as the conditional expected payoff of any other strategy e_k :

$$\sum_j \frac{z_{ij}}{\sum_\ell z_{i\ell}} A_{ij} \geq \sum_j \frac{z_{ij}}{\sum_\ell z_{i\ell}} A_{kj}$$

for every alternative strategy e_k . This means:

$$\sum_j z_{ij} [A_{ij} - A_{kj}] \geq 0 \quad \text{for all } k \neq i.$$

For the other player we find that

$$\sum_i z_{ij} [B_{ij} - B_{i\ell}] \geq 0 \quad \text{for all } \ell \neq j.$$

If we add the (in)equalities

$$z_{ij} \geq 0 \quad \text{for all pairs } (i, j) \quad \text{and} \quad \sum_{(i,j)} z_{ij} = 1$$

we have a description of the set $\mathcal{Z}(A, B)$ of correlated equilibria of the bimatrix game (A, B) as a compact polyhedral set. So calculating $\mathcal{Z}(A, B)$ is the same as calculating the extreme points of $\mathcal{Z}(A, B)$. In Evangelista and Raghavan (1996) it is proved that each extreme point of a maximal Nash set¹ is an extreme point of $\mathcal{Z}(A, B)$ too.

Lemma 1 (i) *A Nash equilibrium (p, q) defines a correlated equilibrium by $z_{ij} := p_i q_j$.*
(ii) *A correlated equilibrium $Z = (z_{ij})$ is a Nash equilibrium if and only if $z_{ij} z_{k\ell} = z_{i\ell} z_{kj}$ for all strategies i, k for player 1 and all strategies j, ℓ for player 2.*

Proof (i) If (p, q) is a Nash equilibrium, we have, for every i, k and j, ℓ

$$p_i > 0 \quad \text{implies} \quad e_i A q \geq e_k A q \quad \text{and} \quad q_j > 0 \quad \text{implies} \quad p B e_j \geq p B e_\ell.$$

This can be written as

$$p_i \left[\sum_j q_j A_{ij} \right] \geq p_i \left[\sum_j q_j A_{kj} \right] \quad \text{and} \quad q_j \left[\sum_i p_i B_{ij} \right] \geq q_j \left[\sum_i p_i B_{i\ell} \right].$$

These are the inequalities we looked for:

$$\sum_j z_{ij} [A_{ij} - A_{kj}] \geq 0 \quad \text{and} \quad \sum_i z_{ij} [B_{ij} - B_{i\ell}] \geq 0 \quad \text{for all } i, k \text{ and all } j, \ell.$$

(ii) If $Z = (z_{ij})$ is a correlated equilibria satisfying all equalities $z_{ij} z_{k\ell} = z_{i\ell} z_{kj}$, we take an entry (i, j) with $z_{ij} \neq 0$ and define the strategies p and q by

$$p_k := \frac{z_{kj}}{\sum_r z_{rj}} \quad \text{and} \quad q_\ell := \frac{z_{i\ell}}{\sum_s z_{is}}.$$

Clearly, p and q are well-defined strategies for player 1 and 2, respectively (the denominators are not zero). We have

$$p_k q_\ell = \frac{z_{kj} z_{i\ell}}{(\sum_r z_{rj})(\sum_s z_{is})} = z_{k\ell} \frac{z_{ij}}{(\sum_r z_{rj})(\sum_s z_{is})}.$$

Every product $p_k q_\ell$ is a product of $z_{k\ell}$ and a constant (since i and j are fixed). This constant is one, since $p_k q_\ell$ as well as $z_{k\ell}$ add up to one. Then $p_k q_\ell = z_{k\ell}$ for all k and ℓ and the Nash equilibrium conditions follow from the conditions for correlated equilibria e.g.:

$$p_i [e_i A q - e_k A q] = \sum_j (A_{ij} - A_{kj}) z_{ij} \geq 0.$$

This completes the proof. \triangleleft

The following lemma will help in the analysis of the next section.

¹A Nash subset is a subset of the set of Nash equilibria with the exchangeability property. If a Nash set is maximal with respect to inclusion, it is a maximal Nash set.

Lemma 2 *The set of correlated equilibria of a bimatrix game (A, B) does not change if the A -matrix is multiplied with a positive factor or a fixed row vector is added to all rows of A . The B -matrix can also be multiplied with any positive factor and any fixed column vector can be added to all columns of B without changing the set of correlated equilibria.*

Proof It is easy to see that the operations proposed give an equivalent system of linear inequalities with the same solution set. \triangleleft

By use of the transformations proposed in Lemma 2 we are able to transform bimatrix games (A, B) into games (A', B') which are strategically equivalent, that is the best reply correspondences and therefore for instance the set of Nash equilibria do not change.

3 Correlated Equilibria for 2×2 -bimatrix games.

In this section we only consider 2×2 -bimatrix games. The payoff matrices are

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

By using Lemma 2 we can transform these matrices into the matrices (A', B') with the same set of correlated equilibria:

$$A' = \begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} \beta_1 & 0 \\ \beta_2 & 0 \end{bmatrix}$$

where $\alpha_i = a_{1i} - a_{2i}$ and $\beta_j = b_{j1} - b_{j2}$. The inequalities describing the set $\mathcal{Z}(A, B) = \mathcal{Z}(A', B')$ are

$$\alpha_1 z_{11} + \alpha_2 z_{12} \geq 0 \tag{1}$$

$$-\alpha_1 z_{21} - \alpha_2 z_{22} \geq 0 \tag{2}$$

$$\beta_1 z_{11} + \beta_2 z_{21} \geq 0 \tag{3}$$

$$-\beta_1 z_{12} - \beta_2 z_{22} \geq 0 \tag{4}$$

We know that the convex polyhedron $\mathcal{Z}(A', B')$ contains the convex hull of the set of Nash equilibria $\mathcal{E}(A', B')$. To compute the set $\mathcal{Z}(A', B')$ it is sufficient to find all (extreme) points of $\mathcal{Z}(A', B')$ that are not Nash equilibria, i.e. $Z \in \mathcal{Z}(A', B')$ with $D(Z) := z_{11}z_{22} - z_{12}z_{21} \neq 0$ (see Lemma 1 (ii)).

Proposition 3 *If $\mathcal{Z}(A', B')$ contains an element Z with $D(Z) \neq 0$, then (A', B') has at least two pure Nash equilibria.*

Proof Let Z be a correlated equilibrium with $D(Z) \neq 0$. If we eliminate α_2 from the inequalities (1) and (2), i.e. multiply inequality (1) with z_{22} and inequality (2) with z_{12} and add we get $\alpha_1 D(Z) \geq 0$. Eliminating α_1 gives $\alpha_2 D(Z) \leq 0$. Eliminating β_2 and β_1 from the inequalities (3) and (4) gives $\beta_1 D(Z) \geq 0$ and $\beta_2 D(Z) \leq 0$.

If $D(Z) > 0$, we find $\alpha_1, \beta_1 \geq 0$ and $\alpha_2, \beta_2 \leq 0$. Then (e_1, e_1) and (e_2, e_2) are pure Nash equilibria. In case $D(Z) < 0$ we find that (e_1, e_2) and (e_2, e_1) are pure Nash equilibria. \triangleleft

Corollary 4 *If the bimatrix game (A', B') has exactly one, completely mixed Nash equilibrium or if one of the players has a strictly dominant strategy, then $\mathcal{Z}(A', B') = \mathcal{E}(A', B')$.*

Remark If there is a correlated equilibrium Z with $D(Z) \neq 0$, we may assume that $D(Z) > 0$. Otherwise we interchange the strategies of player 1 and get

$$A' = \begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 0 & 0 \\ \alpha_1 & \alpha_2 \end{bmatrix}$$

and subtracting the second row from both rows yields

$$\begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 0 & 0 \end{bmatrix}$$

The matrix B' transforms from

$$\begin{bmatrix} \beta_1 & 0 \\ \beta_2 & 0 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} \beta_2 & 0 \\ \beta_1 & 0 \end{bmatrix}.$$

So we assume from this moment that $\alpha_1, \beta_1 \geq 0$ and $\alpha_2, \beta_2 \leq 0$. The strategy pairs (e_1, e_1) and (e_2, e_2) are Nash equilibria and all matrices Z with $z_{12} = z_{21} = 0$ are correlated equilibria. We write $a := \alpha_1$, $a' := -\alpha_2$, $b := \beta_1$ and $b' := -\beta_2$. Then a, b, a' and $b' \geq 0$ and the payoff matrices have the form

$$A'' = \begin{bmatrix} a & -a' \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B'' = \begin{bmatrix} b & 0 \\ -b' & 0 \end{bmatrix}.$$

The inequalities describing the set $\mathcal{Z}(A'', B'')$ are

$$az_{11} - a'z_{12} \geq 0 \tag{1}'$$

$$-az_{21} + a'z_{22} \geq 0 \tag{2}'$$

$$bz_{11} - b'z_{21} \geq 0 \tag{3}'$$

$$-bz_{12} + b'z_{22} \geq 0 \tag{4}'$$

We first consider the case that $a + a' = 0$ or $b + b' = 0$.

Proposition 5 *If $a + a' = 0$ or $b + b' = 0$, then $\mathcal{Z}(A'', B'')$ is the convex hull of $\mathcal{E}(A'', B'')$.*

Proof If $a + a' = 0$ and $b + b' = 0$, both matrices are the 0-matrix and every strategy profile is a Nash equilibrium. Then clearly, $\mathcal{Z}(A'', B'') = \text{ch } \mathcal{E}(A'', B'')$. Next, we consider the case $a + a' = 0$ and $b + b' > 0$. The case $a + a' > 0$ and $b + b' = 0$ is completely similar. By multiplication of the B'' -matrix with a positive factor we get $b + b' = 1$. The following matrices are *Nash equilibria* in the game $(A'' = 0, B'')$:

$$E_{11} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{22} := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Z_1 := \begin{bmatrix} 0 & b' \\ 0 & b \end{bmatrix}, \quad Z_2 := \begin{bmatrix} b' & 0 \\ b & 0 \end{bmatrix}.$$

Let Z be any correlated equilibrium with $z_{12} + z_{21} > 0$. We look for numbers $u_1 \geq 0$ and $u_2 \geq 0$ such that $Z - u_1 Z_1 - u_2 Z_2$ has zeroes off the diagonal and nonnegative diagonal entries. Then we must have $z_{12} = u_1 b'$ and $z_{21} = u_2 b$.

If $b' = 0$, we have by inequality (4)' that $z_{12} = 0$ and if $b = 0$, we also have $z_{21} = 0$ by relation (3)'. So we take $u_1 := \frac{z_{12}}{b'}$ and $u_2 := \frac{z_{21}}{b}$ in as far as these fractions are well-defined and zero else. Also from (3)' and (4)' follows that

$$v_1 := z_{11} - u_2 b' = z_{11} - \frac{z_{21} b'}{b} \geq 0 \quad \text{and} \quad v_2 := z_{22} - u_1 b = z_{22} - \frac{z_{12} b}{b'} \geq 0.$$

Accordingly we find $Z = v_1 E_{11} + v_2 E_{22} + u_1 Z_1 + u_2 Z_2$. \triangleleft

Note that up to this moment we did not find any *extreme point* of $\mathcal{Z}(A'', B'')$ that is not a Nash equilibrium. Only the last class of bimatrix games with $a + a' > 0$ and $b + b' > 0$ can provide us with such examples. We assume that $a + a' = 1$ and $b + b' = 1$.

Consider the following matrices

$$E_{11} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{22} := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$Z_0 := \begin{bmatrix} a'b' & ab' \\ a'b & ab \end{bmatrix}, \quad Z_1 := \begin{bmatrix} a'b' & ab' \\ 0 & ab \end{bmatrix}, \quad Z_2 := \begin{bmatrix} a'b' & 0 \\ a'b & ab \end{bmatrix}.$$

Note that E_{11} , E_{22} and Z_0 are Nash equilibria. Z_0 corresponds with the Nash equilibrium $p = (b', b)$ and $q = (a', a)$.

Proposition 6 *If $a + a' > 0$ and $b + b' > 0$, then*

$$\mathcal{Z}(A'', B'') = \mathbb{R}_+ [E_{11}, E_{22}, Z_0, Z_1, Z_2] \cap \{Z \mid z_{11} + z_{12} + z_{21} + z_{22} = 1\}$$

(the intersection of the positive cone generated by E_{11} , E_{22} , Z_0 , Z_1 and Z_2 and the hyperplane $z_{11} + z_{12} + z_{21} + z_{22} = 1$).

Proof Again we assume that $a + a' = 1$ and $b + b' = 1$. Let Z be a correlated equilibrium with $z_{12} + z_{21} > 0$. We look for numbers u_0, u_1 and $u_2 \geq 0$ such that $Z - u_0 Z_0 - u_1 Z_1 - u_2 Z_2$ has zeroes off the diagonal and nonnegative entries on the diagonal. We have to solve the equations $z_{12} = (u_0 + u_1)ab'$ and $z_{21} = (u_0 + u_2)a'b$.

If $ab' = 0$, we have $a = 0$ and $a' = 1$ or $b' = 0$ and $b = 1$. Then $z_{12} = 0$, in the first case by (1)' and in the second case by (4)'. If $a'b = 0$ we find that $z_{21} = 0$.

We define $u_0 = u_1 = 0$ if $ab' = 0$ and $u_0 = u_2 = 0$ if $a'b = 0$. In other cases we take

$$u_0 := \min \left\{ \frac{z_{12}}{ab'}, \frac{z_{21}}{a'b} \right\}, \quad u_1 := \frac{z_{12}}{ab'} - u_0 \quad \text{and} \quad u_2 := \frac{z_{21}}{a'b} - u_0.$$

Then u_0, u_1 and $u_2 \geq 0$ and $u_0 + u_1 + u_2 = \max \left\{ \frac{z_{12}}{ab'}, \frac{z_{21}}{a'b} \right\}$. Then it is easy to check that

$$z_{11} - (u_0 + u_1 + u_2)a'b' \geq 0 \quad \text{and} \quad z_{22} - (u_0 + u_1 + u_2)ab \geq 0.$$

This completes the proof. ◁

Remark If any of the numbers a, b, a' or b' vanishes, the matrices Z_1 and Z_2 become correlated equilibria associated with Nash equilibria, and $\mathcal{Z}(A'', B'') = \text{ch } \mathcal{E}(A'', B'')$ once again. So, only if a, b, a' and b' are positive, the set $\mathcal{Z}(A'', B'')$ has *two extreme* points Z'_1 and Z'_2 (the updates of Z_1 and Z_2) that are not Nash equilibria and *three extreme* points corresponding to Nash equilibria. Note that $Z_1 + Z_2 = Z_0 + a'b'E_{11} + abE_{22}$. So none of the generators of the cone are in the cone generated by the remaining elements. This implies that all matrices E_{11}, E_{22}, Z_0, Z_1 and Z_2 are extreme directions of the cone and that $E_{11}, E_{22}, Z_0, Z'_1$ and Z'_2 are indeed the extreme points of $\mathcal{Z}(A'', B'')$.

Example Consider the bimatrix game with payoff matrices:

$$A := \begin{bmatrix} 13 & 9 \\ 6 & 11 \end{bmatrix}, \quad B := \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}.$$

Then

$$A' := \begin{bmatrix} 7 & -2 \\ 0 & 0 \end{bmatrix}, \quad B' := \begin{bmatrix} 6 & 0 \\ -4 & 0 \end{bmatrix}$$

and finally

$$A'' := \begin{bmatrix} \frac{7}{9} & -\frac{2}{9} \\ 0 & 0 \end{bmatrix}, \quad B'' := \begin{bmatrix} \frac{6}{10} & 0 \\ -\frac{4}{10} & 0 \end{bmatrix}.$$

The matrix Z_0 equals

$$Z_0 = \begin{bmatrix} \frac{8}{90} & \frac{28}{90} \\ \frac{12}{90} & \frac{42}{90} \end{bmatrix}.$$

The ‘non-Nash’ extreme points of $\mathcal{Z}(A'', B'')$ and thus those of $\mathcal{Z}(A, B)$ are

$$Z'_1 = \begin{bmatrix} \frac{8}{78} & \frac{28}{78} \\ 0 & \frac{42}{78} \end{bmatrix} \quad \text{and} \quad Z'_2 = \begin{bmatrix} \frac{8}{62} & 0 \\ \frac{12}{62} & \frac{42}{62} \end{bmatrix}.$$

Conclusion For almost all 2×2 -bimatrix games the set of correlated equilibria equals the convex hull of the equilibrium set. Only if the game has three isolated equilibria the set of extreme points of $\mathcal{Z}(A, B)$ consists of the *three* Nash equilibria and *two* additional points, not corresponding to a Nash equilibrium.

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